mathleague.org

Solutions
Round 11614

Sprint Test

1. 110
2. 17
3. 105
4. 38
5. 6
6. 10
7. 260
8. 8060π
9. 41
10. 271
11. 123
12. 6
13. 96
14. 20
15. 4032
16. $\frac{169}{425}$
17. 72
18. 24
19. 26
20. 98
21. $\frac{3}{4}$
22. $\frac{27}{16}$
23. $-\frac{3}{2}$
24. 52
25. $8+24\sqrt{3}$
26. 1111
27. 403
28. 172
29. 0
30. $\frac{2575}{10302}$

Target Test

1. 6
2. 60
3. 78
4. 16
5. $\frac{29}{99}$
6. 441
7. 131
8. 96

Team Test

1. 6
2. 13
3. 33
4. 1296
5. 720
6. -21
7. 15
8. 1
9. 10102
10. 4088

mathleague.org

Solutions
Round 11614

Sprint Test Solutions

1. $\frac{75 \text{ miles}}{1 \text{ hour}} \cdot \frac{1 \text{ hour}}{3600 \text{ seconds}} \cdot \frac{5280 \text{ feet}}{1 \text{ mile}} = \frac{110 \text{ feet}}{1 \text{ second}}$. Factoring and cancelling is highly recommended.

2. There are 14 squares less than 200 (1^2 through 14^2) and 5 cubes less than 200 (1^3 through 5^3). But two of these numbers (1 and 64) are both squares and cubes, so they should only be counted once. Thus when we add 14 and 5, we need to subtract 2 for the 2 numbers that were double counted. This gives us $14 + 5 - 2 = 17$ eligible numbers.

3. First, take the prime factorizations of the numbers: $42 = 2^1 \cdot 3^1 \cdot 7^1$ and $90 = 2^1 \cdot 3^2 \cdot 5^1$. The LCM can be found by multiplying the highest powers of any prime factors that appear in either number: $2^1 \cdot 3^2 \cdot 5^1 \cdot 7^1$. Don't bother multiplying this, because a lot of it will cancel with the GCD. The GCD can be found by multiplying the lowest powers of any prime factors that appear in both numbers: $2^1 \cdot 3^1$. Now $\frac{\text{LCM}(42,90)}{\text{GCD}(42,90)} = \frac{2^1 \cdot 3^2 \cdot 5^1 \cdot 7^1}{2^1 \cdot 3^1} = 3 \cdot 5 \cdot 7 = 105$.

4. $5 \ominus 3 = 5^2 \cdot 3 - 3^2 = 66$, and $4 \ominus 2 = 4^2 \cdot 2 - 2^2 = 28$. $66 - 28 = 38$.

5. The equation for this is $(b+3) = 3(b-3)$. Solving, we get $b = 6$.

6. B is diametrically opposite A, so double the radius is 10.

7. Dawn is 240 meters east and 100 meters south of her starting point. By the Pythagorean Theorem, she is c meters from her starting point, where $240^2 + 100^2 = c^2$. We can solve for c from this equation or notice that the triangle we're dealing with is similar to a 5-12-13 triangle, scaled by a factor of 20. Either way, we get $c = 260$.

8. The area of the annulus (the region between the two circles) is the difference of the areas. Thus the area is $\pi 2016^2 - \pi 2014^2 = \pi(2016 + 2014)(2016 - 2014) = \pi(4030)(2) = 8060\pi$.

9. By the distance formula, this is $\sqrt{(-7-2)^2 + (-16-24)^2} = \sqrt{81 + 1600} = 41$.

10. The sum of the first nine terms is 729, and the sum of the first ten terms is 1000, so the tenth term must be the difference between those two sums. $1000 - 729 = 271$.

11. For each leg, divide the number of miles by the number of miles per hour to get the number of hours for that leg: $\frac{40}{50} + \frac{50}{40} = \frac{41}{20}$ hours. Multiply this by 60 to get the number of minutes.

12. The volume of a pyramid is $V = \frac{1}{3}Bh$, so we have $132 = \frac{1}{3}B11$. This means B, the area of the base, is 36, giving us a square with side length 6.

13. Note that $111_2 = 1000_2 - 1_2$, which equals $2^3 - 1$ in base 10. $222_3 = 1000_3 - 1_3$, which equals $3^3 - 1$ in base 10. $333_4 = 1000_4 - 1_4$, which equals $4^3 - 1$ in base 10. Adding these together we get $2^3 + 3^3 + 4^3 - 3 = 96$.

14. Suppose the polygon has n sides. Then the sum of the measures of the interior angles is $180(n-2)$. But we are given in the problem that this sum is $170(n-1) + 10$. So we set these equal to get $180n - 360 = 170n - 160$. Solving for n we get $n = 20$.

15. Converting these fractions to decimals, $\frac{9}{11} = 0.\overline{81}$ and $\frac{11}{9} = 1.\overline{2}$. Adding these, we get $2.\overline{04}$. Another way to do this is to note that $\frac{11}{9} + \frac{9}{11} = \frac{202}{99} = 2\frac{4}{99} = 2.\overline{04}$. Either way, every other digit is a 4, and there will be 1008 of these 4s in the first 2016 digits after the decimal point for a total of 4032.

16. The first card can be of any suit. After that, there are 51 cards left in the deck and 39 of a different suit from the first one. After choosing one of these, there are 50 cards in the deck and 26 of a different suit from either of the first two cards. Thus the probability is $\frac{39}{51} \cdot \frac{26}{50} = \frac{169}{425}$. Alternatively, there are $\binom{52}{3} = 22100$ ways to choose three cards from the deck. In order to choose three cards of different suits, we have

to choose one suit (4 choices) to be left out and then one card (13 choices) from each suit we are using, for a total of $4 \cdot 13^3$ successful outcomes. The answer is then $\frac{4 \cdot 13^3}{22100} = \frac{169}{425}$.

17. The four vertices of this square are going to be the minimum of the first graph, the maximum of the second, and the two points of intersection of the two graphs. The minimum of the first graph is at the point $(-4, -3)$, and the maximum of the second graph is at the point $(-4, 9)$. To find one intersection point of the two graphs, just remove the absolute values and set the two functions equal to each other: $x + 4 - 3 = -x - 4 + 9$, so $x = 2$ and the intersection is at $(2, 3)$. To find the second intersection, multiply what's inside the absolute value signs by -1 and remove the absolute value signs: $-x - 7 = x + 13$, so $x = -10$ and the intersection is at $(-10, 3)$. We now have the four vertices of a square with side length $6\sqrt{2}$. Square this side length to get the area.

18. This is $(100 - 1)(100 - 2)(100 - 3)(100 - 4)$. When distributed, all terms are a multiple of 100 except for the $(-1)(-2)(-3)(-4)$ term, which evaluates to 24.

19. This quadrilateral is a square with side length $\sqrt{26}$, and thus its area is 26.

20. The interior diagonal of a cube is $\sqrt{3}$ times the edge length, so the edge length is $\frac{7}{\sqrt{3}}$. Each face then has area $(\frac{7}{\sqrt{3}})^2 = \frac{49}{3}$. Multiply this by 6 to get the total surface area.

21. You could find the lowest common denominator, but here's a prettier solution: Rewrite the sum as $2\left(\frac{1}{6} + \frac{1}{12} + \ldots + \frac{1}{56}\right)$, and then use the fact that $\frac{1}{6} = \frac{1}{2} - \frac{1}{3}$, $\frac{1}{12} = \frac{1}{3} - \frac{1}{4}$, and so on. The sum becomes $2\left(\frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \ldots + \frac{1}{7} - \frac{1}{8}\right)$, or $2\left(\frac{1}{2} - \frac{1}{8}\right) = \frac{3}{4}$.

22. Let the square and triangle have perimeter $12x$. A side length of the square is $3x$, and thus the radius of the inscribed circle has length $\frac{3x}{2}$. Its area is $\frac{9\pi x^2}{4}$. The triangle has side length $4x$; by dropping an altitude and using $30-60-90$ right triangle properties, we determine that the inradius of the triangle is $\frac{2x\sqrt{3}}{3}$ (or by using $A = r \cdot s$). The area of this circle therefore is $\frac{4x^2\pi}{3}$, and our desired ratio of areas is $\frac{\frac{9x^2\pi}{4}}{\frac{4x^2\pi}{3}}$ or $\frac{27}{16}$.

23. By Vieta's formulas (which, if you haven't seen them, relate the coefficients of a polynomial to things like the sum and product of the roots), the sum of the roots is $-\frac{1}{2}$. This means the last root is $-\frac{1}{2} - (3 - 2) = -\frac{3}{2}$.

24. The centers of these circles are $(-13, -4)$ and $(2, 16)$, so the distance between the centers is $\sqrt{(2+13)^2 + (16+4)^2} = \sqrt{625} = 25$. The distance from the center of the first circle to any point on the circle is the radius, which is 12. Similarly, the distance from the center of the second circle to any point on the circle is 15. The maximum distance between any two points on these circles is then $25 + 12 + 15 = 52$.

25. If $4^x + 4 = 16$, then $12 = 4^x = (2^x)^2$ and $2^x = \sqrt{12} = 2\sqrt{3}$. Now $8^x = (2^x)^3 = 8\sqrt{27} = 24\sqrt{3}$, so the answer is $8 + 24\sqrt{3}$.

26. This number breaks down into $1111^2 = 11^2 \cdot 101^2$, so the distinct prime factors are 11 and 101. $11 \cdot 101 = 1111$. For future reference, note that $11^2 = 121$, $111^2 = 12321$, $11111^2 = 123454321$, and so on.

27. Because we are looking for lattice points, both x and y must be integers. Solving for y, we see that $y = \frac{6-4x}{5}$ must be an integer. Thus we need $6 - 4x$ to be divisible by 5. The first positive integer x for which this is true is $x = 4$, and after that every time we add 5 to x we also get $6 - 4x$ to be divisible by 5. Now we just need to count the number of terms in the set $\{4, 9, 14, \ldots, 2014\}$. This is the same as the number of terms in $\{5, 10, 15, \ldots, 2015\}$ (we just shifted the numbers up by 1) and in $\{1, 2, 3, \ldots, 403\}$ (we just divided the numbers

by 5).

28. At 5:00, the angle formed by the hands of the clock comprises $\frac{5}{12}$ of a circle, or 150°. Every minute, the minute hand moves $\frac{1}{60}$ of the way around the circle, or 6°, and the hour hand moves $\frac{1}{12}$ as much as the minute hand, or 0.5°. This means every minute the angle between the hands on the clock changes by 5.5°. Four minutes prior to 5:00, the hands would have been 22° further apart then they were at 5:00.

29. Factoring by the difference of squares, we get $442(442^2+1)(442+1)(442-1)$. The first term is divisible by 2, the second term is divisible by 5, and the last term is divisble by 3. Thus, the remainder when divided by $2 \cdot 3 \cdot 5$ is 0.

30. Let's try to express $f(x)$ in terms of two telescoping sequences. $\frac{1}{x(x+1)} - \frac{1}{(x+1)(x+2)} = \frac{2}{x(x+1)(x+2)}$, so we want the sum of $\frac{1}{x(x+1)}$ from 1 to 100 minus the sum of $\frac{1}{x(x+1)}$ from 2 to 101, with the difference divided by two. All the terms except for the first and the last cancel out, so we simply evaluate $\frac{\frac{1}{1(2)} - \frac{1}{101(102)}}{2}$, which is $\frac{2575}{10302}$.

Target Test Solutions

1. B, C, and D can be visited in any order, giving $3! = 6$ possibilites.
2. There are $\binom{6}{4}$ ways to choose the squad, then 4 ways to choose the benchwarmer. Multiplying yields 60 total ways.
3. Adding up squares one by one, we see that $1+4+9+16+25+36 = 91$, so $k = 6$. The sum of the first $2k = 12$ positive integers is $\frac{12 \cdot 13}{2} = 78$.
4. We are looking for the number of integers from 1 to 60 that are relatively prime to 60 (these will be the numerators of the fractions). Half of them are out as multiples of 2, one third of the remaining numbers are out as multiples of 3, and one fifth of the remaining numbers are out as multiples of 5. This gives us $60 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = 16$.
5. The digits which are divisible by 4 are 0, 4, and 8. We can obtain a units digit of 0 from the 9 positive integers less than 100 which end with 0, and can obtain a units digit of 4 from the 10 ending in 2 and the 10 ending in 8, for a total of 29 of the 99 positive integers less than 100.
6. Choosing four points in a rectangle is equivalent to choosing two rows and two coloumns, and using the 4 intersections as vertices of a rectangle. This gives $\binom{7}{2}^2$ ways to choose points in a rectangle, or $21^2 = 441$ total ways.
7. It may be best to list the terms of the sequence and cross out the ones that are not prime. 1 is not prime by definition. 21, 51, 81, and 111 are multiples of 3. 91 is a multiple of 7. 121 is a multiple of 11. This leaves 11, 31, 41, 61, 71, 101, and 131 as the first seven primes.
8. In order to have the shortest possible distance, $\overline{AB}$ must be a "mutual internal tangent" of the two circles, i.e. the two circles are on opposite sides of $\overline{AB}$. Let O and P be the centers of the two circles, with $OA = 16$ and $BP = 12$. Drop a perpendicular from P to OA, intersecting the extension of OA at C. Then $ABPC$ is a rectangle, so $AC = 12$. Now OQC is a right triangle with $OQ = 100$ and $OC = 28$, so $QC = AB = 96$. (A similar calculation shows that if AB is a mutual *external* tangent, then $AB = \sqrt{100^2 - 4^2} > 96$.

Team Test Solutions

1. Using the binomial theorem, $11^{2016} = (10+1)^{2016} = \sum_{i=0}^{2016} 1^i \cdot 10^{2016-i}\binom{2016}{i}$. The only terms that are not divisible by 100 (and therefore irrelevant) are the last two, so we need only find the tens digit of $10 \cdot \binom{2016}{2015} + \binom{2016}{2016} = 20160 + 1$, which is 6. Of course, you could figure out the answer with a much less rigorous approach just by taking the first few powers of 11 and observing the pattern.

Solutions
Round 11614

2. The longest distance of BD is obtained when D, A, C, and B are in that order on a line. This yields a length of $3 + 5 + 5 = 13$.

3. We are looking for the smallest positive integer that is 3 more than a multiple of 6 and 5 more than a multiple of 7. Listing out the possibilities, the first set contains $\{3, 9, 15, 21, 27, 33, 39, ...\}$ and the second set contains $\{5, 12, 19, 26, 33, 40, ...\}$. The smallest number that appears in both sets is 33.

4. Call the second term x. Then the third term is $\frac{2}{3}x$, the fourth term is $\frac{4}{9}x$, and the fifth term is $\frac{8}{27}x$. Since we are given that the fifth term is 384, we have $384 = \frac{8}{27}x$, so $x = \frac{384 \cdot 27}{8} = 48 \cdot 27 = 1296$.

5. There are 900 three-digit positive integers. We need to subtract from that the number of three-digit positive integers that have no prime digits. There are 5 choices for the first digit (1, 4, 6, 8, 9), 6 choices for the second digit (0, 1, 4, 6, 8, 9), and 6 choices for the third digit (0, 1, 4, 6, 8, 9). This gives us $5 \cdot 6 \cdot 6 = 180$ integers with no prime digits. Subtracting this from 900 gives us 720.

6. We need to set the two functions equal to each other and choose k to make sure there is only one solution. If $2x^2 + 7x - 5 = x^2 - x + k$, then $x^2 + 8k - 5 - k = 0$. In order to have only one solution, the discriminant of this quadratic must be 0, so $b^2 - 4ac = 8^2 - 4(-5 - k) = 0$. This gives us $64 = 4(-5 - k)$ and $5 + k = -16$, so $k = -21$.

7. Triangle ACD is similar to triangle CBD, so $\frac{AD}{DC} = \frac{DC}{BD}$. Plugging in 9 for AD and 25 for BD, we get $DC = 15$.

8. Let the incircle have center I, and the circle of radius r have center O. Let the tangency point of the incircle to AB be D, and let the tangency point from circle O to AB be E. DIA is a $30 - 60 - 90$ triangle, so AI is 6 and AO is $3 - r$. AEO is also a $30 - 60 - 90$ triangle with $EO = r$, so $3 - r = 2r$, and $r = 1$.

9. First, $729,000 = 2^3 \cdot 3^6 \cdot 5^3$. For a factor to be an odd perfect cube, the exponent of the 2 must be 0, the exponent of the 3 must be 0, 3, or 6, and the exponent of the 5 must be 0 or 3. The sum of these factors is equivalent to the expansion of the expression $(2^0)(3^0+3^3+3^6)(5^0+5^3) = 1 \cdot 757 \cdot 126 = 95382$. Similarly, the sum of the even perfect squares is $(2^2)(3^0+3^2+3^4+3^6)(5^0+5^2) = 4 \cdot 820 \cdot 26 = 85280$. $95382 - 85280 = 10102$.

10. Taking $f(n + 8)$ is the same as taking n in binary and adding a 1 to the fourth digit from the right (the "eights place"). If this fourth digit is zero, then it will just become a 1, increasing the sum of digits by 1, so this won't work. If the fourth digit is 1, though, it'll become 0 and carry a 1 over to the next place. If this place is also 1, then it will do the same. If there are many consecutive 1s, they will all become 0s and carry a 1 over to the next digit to the left. If n has k consecutive 1s starting from the fourth digit, then its binary representation is (first few digits) (0) (k consecutive 1s) (three digits). This number will become (first few digits) (1) (k consecutive 0s) (three digits). As we're subtracting k 1's, and adding one 1, the sum of the digits will decrease by $k - 1$. We then have $f(n) - f(n+8) = k - 1$, and we want $k - 1 = 8$, so $k = 9$. Our number then has 9 consecutive 1's. To minimize the value, we make $n = 111111111000_2$, or in base 10, $n = 4088$.